

# AN EQUIVALENT FORM OF YOUNG'S INEQUALITY WITH UPPER BOUND

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Young's integral inequality is complemented with an upper bound to the remainder. The new inequality turns out to be equivalent to Young's inequality, and the cases in which the equality holds become particularly transparent in the new formulation.

## 1. Formulation of the theorem

Let  $\phi : [\alpha_1, \alpha_2] \rightarrow [\beta_1, \beta_2]$  be a continuous increasing function and let  $\psi : [\beta_1, \beta_2] \rightarrow [\alpha_1, \alpha_2]$  be its inverse,  $\psi(\phi(\alpha)) = \alpha$  (so that  $\phi(\alpha_i) = \beta_i$ ,  $i = 1, 2$ ). Define

$$(1) \quad F(a, b) = \int_{\alpha_1}^a \phi \, dx + \int_{\beta_1}^b \psi \, dx - ab + \alpha_1 \beta_1.$$

Young's inequality [2, 1, 4] states that for every  $a \in [\alpha_1, \alpha_2]$  and  $b \in [\beta_1, \beta_2]$

$$(2) \quad 0 \leq F(a, b),$$

where the equality holds iff  $\phi(a) = b$  (or, equivalently,  $\psi(b) = a$ ).

Among the classical inequalities Young's inequality is probably the most intuitive. Indeed, its meaning can be easily grasped once the integrals are regarded as areas below and on the left of the graph of  $\phi$  (see, for instance, [5]). Despite its simplicity, it has profound consequences. For instance, the Cauchy, Holder and Minkowski inequalities can be easily derived from it [5].

In this work I am going to improve Young's inequality as follows

**THEOREM 1.1.** *Under the assumptions of Young's inequality, we have for every  $a \in [\alpha_1, \alpha_2]$  and  $b \in [\beta_1, \beta_2]$ ,*

$$(3) \quad 0 \leq F(a, b) \leq -(\psi(b) - a)(\phi(a) - b).$$

*where the former equality holds if and only if the latter equality holds.*

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2000 Mathematics Subject Classification: 26D10, 44A15  
Keywords and Phrases: Young's inequality, Legendre transform.

Note that the theorem contains Young's inequality as a special case, with the advantage that the equality case is naturally taken into account by the special form of the upper bound. For instance, if  $\psi(b) = a$  then  $F(a, b) = 0$  which is one of the additional statements contained in the classical formulation of Young's inequality. Nevertheless, I will not prove again Young's inequality, instead I will use it repeatedly to obtain the extended version given by theorem 1.1.

REMARK 1.2. Over the years several extensions of Young's inequality have been considered. A good account is given by [4]. Among those only M. Merkle's contribution [3] seems to go in the same direction considered by this work. Theorem 1.1 improves Merkle's result, which in the case  $\alpha_1 = \beta_1 = 0$  states that (notation of this work)

$$F(a, b) \leq \max\{a\phi(a), b\psi(b)\} - ab.$$

Indeed, the last term of (3) can be rewritten

$$[a\phi(a) + b\psi(b) - \phi(a)\psi(b)] - ab,$$

and we have only to show that

$$a\phi(a) + b\psi(b) - \phi(a)\psi(b) \leq \max\{a\phi(a), b\psi(b)\},$$

and that for some  $a, b$ , the inequality is strict. Indeed, if  $\phi(a) > b$  then, since  $\phi$  and  $\psi$  are one the inverse of the other,  $a > \psi(b)$  and thus  $a\phi(a) > b\psi(b)$ . Then

$$a\phi(a) + b\psi(b) - \phi(a)\psi(b) = a\phi(a) + (b - \phi(a))\psi(b) < a\phi(a) = \max\{a\phi(a), b\psi(b)\}$$

The case  $\phi(a) < b$  gives again a strict inequality while the case  $\phi(a) = b$  gives an equality.

## 2. The proof

The proof of theorem 1.1 is based on the next lemma

LEMMA 2.1. *For every  $a, \tilde{a} \in [\alpha_1, \alpha_2]$  and  $b, \tilde{b} \in [\beta_1, \beta_2]$ , we have*

$$(4) \quad F(a, b) + F(\tilde{a}, \tilde{b}) \geq -(\tilde{a} - a)(\tilde{b} - b),$$

where the equality holds iff  $\tilde{a} = \psi(b)$  and  $\tilde{b} = \phi(a)$ .

PROOF. Young's inequality gives

$$(5) \quad \int_{\alpha_1}^a \phi \, dx + \int_{\beta_1}^{\tilde{b}} \psi \, dx + \alpha_1 \beta_1 \geq a\tilde{b}$$

$$(6) \quad \int_{\alpha_1}^{\tilde{a}} \phi \, dx + \int_{\beta_1}^b \psi \, dx + \alpha_1 \beta_1 \geq \tilde{a}b$$

then

$$\begin{aligned}
 & [\int_{\alpha_1}^a \phi \, dx + \int_{\beta_1}^b \psi \, dx - ab + \alpha_1 \beta_1] + [\int_{\alpha_1}^{\tilde{a}} \phi \, dx + \int_{\beta_1}^{\tilde{b}} \psi \, dx - \tilde{a}\tilde{b} + \alpha_1 \beta_1] \\
 &= [\int_{\alpha_1}^a \phi \, dx + \int_{\beta_1}^{\tilde{b}} \psi \, dx + \alpha_1 \beta_1] + [\int_{\alpha_1}^{\tilde{a}} \phi \, dx + \int_{\beta_1}^b \psi \, dx + \alpha_1 \beta_1] - ab - \tilde{a}\tilde{b} \\
 &\geq a\tilde{b} + \tilde{a}b - ab - \tilde{a}\tilde{b} = -(\tilde{a} - a)(\tilde{b} - b).
 \end{aligned}$$

The equality holds iff it holds in (5) and (6), that is iff  $\tilde{a} = \psi(b)$  and  $\tilde{b} = \phi(a)$ .  $\square$

We are ready to prove the theorem

PROOF OF THEOREM 1.1. Consider (4) with  $\tilde{a} = \psi(b)$  and  $\tilde{b} = \phi(a)$

$$F(a, b) + F(\psi(b), \phi(a)) = -(\psi(b) - a)(\phi(a) - b).$$

By Young's inequality, since  $\psi(b) \in [\alpha_1, \alpha_2]$  and  $\phi(a) \in [\beta_1, \beta_2]$ ,  $F(\psi(b), \phi(a)) \geq 0$ , thus

$$F(a, b) \leq -(\phi(a) - b)(\psi(b) - a).$$

The equality holds iff  $F(\psi(b), \phi(a)) = 0$  which holds, again by the usual Young's inequality, iff  $\phi(\psi(b)) = \phi(a)$  i.e.  $b = \phi(a)$  (or equivalently  $a = \psi(b)$ ), which holds iff the inequality  $F(a, b) \geq 0$  is actually an equality.  $\square$

### 3. The Legendre transform

It is worthwhile to recall the connection with the Legendre transform. If  $\Phi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$  and  $\Psi : [\beta_1, \beta_2] \rightarrow \mathbb{R}$  are two  $C^1$  functions with increasing derivatives such that they are the Legendre transform of each other then it is well known that they admit the integral representation  $\Phi(a) = \Phi(\alpha_1) + \int_{\alpha_1}^a \phi \, dx$ ,  $\Psi(b) = \Psi(\beta_1) + \int_{\beta_1}^b \psi \, dx$  where  $\phi$  and  $\psi$  are two  $C^0$  increasing function which are one the inverse of the other,  $\beta_1 = \phi(\alpha_1)$  and  $\Phi(\alpha_1) + \Psi(\beta_1) = \alpha_1 \beta_1$ . Thus the theorem for the Legendre transforms case takes the following form

THEOREM 3.1. *If  $\Phi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$  and  $\Psi : [\beta_1, \beta_2] \rightarrow \mathbb{R}$  are two  $C^1$  functions with increasing derivatives such that they are the Legendre transform of each other, then for every  $a \in [\alpha_1, \alpha_2]$ ,  $b \in [\beta_1, \beta_2]$*

$$(7) \quad 0 \leq \Phi(a) + \Psi(b) - ab \leq -(\Phi'(a) - b)(\Psi'(b) - a),$$

where the former equality holds iff the latter equality holds.

EXAMPLE 3.2. Take  $\Phi(a) = \frac{a^\alpha}{\alpha}$  and  $\Psi(b) = \frac{b^\beta}{\beta}$  with  $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ , and  $\alpha, \beta > 1$ , then we obtain the inequalities

$$(8) \quad 0 \leq \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta} - ab \leq -(a^{\alpha-1} - b)(b^{\beta-1} - a),$$

in particular the last inequality can be rewritten

$$b^{\beta-1}a^{\alpha-1} \leq \frac{1}{\alpha} b^{\beta} + \frac{1}{\beta} a^{\alpha} = \frac{1}{\alpha} (b^{\beta-1})^{\alpha} + \frac{1}{\beta} (a^{\alpha-1})^{\beta},$$

that is, it has as expected the same form of Young's inequality.

### **Acknowledgments**

This work has been partially supported by GNFM of INDAM and by MIUR under project PRIN 2005 from Università di Camerino.

## Bibliography

- [1] J. B. DIAZ AND F. T. METCALF: *An analytic proof of Young's inequality*. Amer. Math. Monthly **77** (1970), 603–609.
- [2] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA: “Inequalities”, Cambridge University Press, Cambridge, 1934.
- [3] M. MERKLE: *A contribution to Young's inequality*. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **461–497** (1974) 265–267.
- [4] D. S. MITRINOVIĆ, J. E. PECĀRIĆ, AND A. M. FINK: “Classical and new inequalities in analysis”, Kluwer Academic Publishers, Dordrecht, 1993.
- [5] E. TOLSTED: *An elementary derivation of the Cauchy, Holder, and Minkowski inequalities from Young's inequality*. Math. Mag. **37** (1964) 2–12.

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